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Applying the dynamics of the pendulum to the design of a playground swing

Josué Njock Libii

Indiana University - Purdue University Fort Wayne
Fort Wayne, United States of America

ABSTRACT: In teaching mechanics courses, it is common practice to neglect nonlinear terms that arise in derivations. The reason used to justify this is that such approximations are adequate for small deformations. However, it is seldom clear how small is *small* or how large the maximum error is likely to be. The errors incurred by neglecting such terms are illustrated in this paper by taking the results of the dynamics of the pendulum and applying them to the design of a playground swing in a city park. The dynamic forces that arise during the operation of a swing are determined under the conditions of small angles and under those of large angles. The resulting forces are compared to see the effect of nonlinearity. Forces in the cables that support the seat of the swing are shown to be smaller when computed using the linearised equation of motion than those obtained from the nonlinear equation itself and the discrepancy between the two results increases with the magnitude of the initial angle that is given to the swing. The effect of this discrepancy on design is that the moments and forces used to design the beam and fasteners that hold the seat of the swing in place are underestimated significantly by linearising the driving force. The extent to which these forces are underestimated is quantified.

INTRODUCTION

The motion of a pendulum is studied in many college courses. These include college physics, ordinary differential equations, dynamics, controls, vibrations and acoustics. However, in all these cases, the differential equation that describes this motion is linearised by assuming that the amplitude of oscillation is small. As a consequence, students do not see how the oscillation of a pendulum is affected by large amplitudes of motion; nor do they know the limits of applicability of the linearised solution they have studied. It turns out that, up to angles of 30°, the period of oscillation obtained from the linearised equation is accurate to within 5%. However, in order to achieve this same accuracy, the angles of swing obtained from solving the linearised equation must be kept below 10° [1].

Observations of children in local city parks revealed that a child on a swing can cause it to move through angles that can be as large 90° from the vertical. Thus, to be realistic, the design of a swing requires the consideration of a nonlinear effect because the angles of swing involved exceed those that are required to make the linearisation of the equation of motion valid.

Oscillations of a pendulum that include large amplitudes have been studied for the purpose of comparing them to those for small amplitudes [1]. It is known that in both cases, the angle of a swing is a periodic function of time. For small angular displacements, the period of oscillation is a constant and the ensuing angle of swing can be represented accurately by means of circular functions. However, for large amplitudes, the period is represented by Jacobi's complete elliptic integral of the first kind and varies with the initial amplitude, while the corresponding angle of swing is represented by elliptic functions of Jacobi. It was demonstrated that the period of the linearised motion is always smaller than, or equal to, that from the nonlinear motion and that, as a general rule, it is inaccurate to use the magnitude of the error made in approximating $\sin\theta$ with θ as an estimate of the accuracy on how well the linearised solution approximates the exact motion [1].

In this article, oscillations of a swing are used to compute the forces that arise during the operation of a swing under conditions of small angles and those of large angles. The results are then compared in order to assess the errors induced in the magnitude of the forces by neglecting the nonlinear effects.

THE BASIC EQUATIONS

Consider a rigid body that is suspended from a point O about which it oscillates freely in the vertical plane. Let the angular displacement about the vertical axis be denoted by θ, measured in radians. After applying Newton's second law of motion in polar coordinates, we obtain two equations of motion. Using the equation of motion in the tangential direction, we find that the angle θ can be obtained by solving the equation [1]

$$\ddot{\theta} + \omega_n^2 \sin(\theta) = 0 . \tag{1}$$

In general, the conditions at the starting time, $t = t_s$, are given by [1]:

$$t = t_s, \theta(t_s) \equiv \theta_s, \dot{\theta}(t_s) \equiv \dot{\theta}_s . \tag{1a}$$

In these equations, the dots represent differentiation with respect to time t and the quantity ω_n, which has units of rad/s, is related to the natural frequency of the system.

As an example, for a compound pendulum swinging in the vertical plane about a horizontal axis that goes through point O,

$$\omega_n \equiv \sqrt{\frac{m_{total} g d}{J_0}} \tag{1b}$$

where, m_{total} is the total mass of the pendulum; g is the acceleration of gravity; d is the distance between point O and the centre of mass of the pendulum; and J_0 is the (polar) mass moment of inertia of the body about point O. It can be seen that ω_n is a physical parameter that is independent of time [1].

By applying Newton's second law of motion in the radial direction, the force of tension in each cable that connects the seat of the swing to its support is found to be given by:

$$T = \frac{mg}{2}(\cos\theta + \frac{1}{\omega_n^2}\dot{\theta}^2)$$

When the pendulum is at rest, each tension is equal to half the weight of the child using the swing. In order to be able to compare the dynamic tension to this static tension, one uses a dimensionless ratio of the two forces, as shown below.

$$\frac{T}{\left(\frac{mg}{2}\right)} = \cos\theta + \left(\frac{\dot{\theta}}{\omega_n}\right)^2 \tag{1c}$$

The objective is to solve Eq. (1) by assuming small angular displacements and again by assuming displacements of arbitrary size. Then, by using the results so obtained to compute and compare how the force of tension expressed in Eq. (1c) is affected by the magnitude of the initial displacement of the pendulum, one obtains a means for determining the effect of linearising the differential equation on the force that is actually applied to the supports that hold the swing in place.

THE CASE OF SMALL ANGLES OF SWING

For small amplitudes, it is conventional to linearise Eq. (1) by expanding the sin θ into a power series as shown below.

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots + \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + \cdots \tag{2}$$

and replacing the sin θ with θ, the first term in that series.

Doing so gives:

$$\ddot{\theta} + \omega_n^2 \theta = 0 \tag{3}$$

This is the equation that is used in all the courses mentioned above**.** Its solution is:

$$\theta(t) = A\sin(\omega_n t) + B\cos(\omega_n t) \tag{4}$$

In this case, ω_n is the circular frequency of the motion expressed in radians per second.

After the initial conditions given in Eq. (1a) are used in Eq. (4), the constants A and B are found to be given, respectively, by:

$$A = \theta_s \sin(\omega_n t_s) + \tfrac{\dot{\theta}_s}{\omega_n}\cos(\omega_n t_s)$$
$$B = \theta_s \cos(\omega_n t_s) - \tfrac{\dot{\theta}_s}{\omega_n}\sin(\omega_n t_s) \tag{5}$$

However, in order to obtain a solution with a simple mathematical form, it is conventional to let α be the maximum amplitude of oscillation and set $t_s \equiv 0, \theta_s \equiv 0, \dot{\theta}_s \equiv \omega_n \alpha$. Incorporating these assumptions into Eq. (5) leads to $A = \alpha$ and $B = 0$; and Eq. (4) becomes:

$$\theta(t) = \alpha \sin(\omega_n t) \;. \tag{6}$$

Here, the period of oscillation, τ_n, is related to the circular frequency, ω_n, by:

$$\tau_n = \frac{2\pi}{\omega_n} \tag{6a}$$

Using Eq. (6), the tension in the swing cable expressed in Eq. (1c) becomes:

$$\frac{T}{\left(\frac{mg}{2}\right)} = \cos\theta(1 + \alpha^2 \cos\theta) \cdot \tag{6b}$$

THE GENERAL CASE OF ANY ANGLE

When swinging angles may be large, Eq. (1) is transformed into Jacobi's elliptic integral of the first kind by two successive integrations and a change of variables [1]. The exact solution to Eq. (1) is found to be [1]:

$$\theta(t) = 2\,Arc\sin\left[\sin\left(\frac{\alpha}{2}\right) sn(\omega t)\right], \tag{7}$$

where sn(ωt) denotes the sine amplitude of ωt, a Jacobi's elliptic function with the elliptic modulus suppressed [4-9].

The elliptic functions of Jacobi are defined as inverses of Jacobi's elliptic integral of the first kind [4-9]. Thus, if one writes:

$$u = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}},$$

then, for example, the sine amplitude $sn(u,k) = \sin(\phi)$, the cosine amplitude is $cn(u,k) = \cos(\phi)$, and the delta amplitude is $dn(u,k) = \sqrt{1 - k^2 \sin^2(\phi)}$, where the parameter k is related to the maximum angle of swing by $k^2 \equiv \sin^2\left(\frac{\alpha}{2}\right)$.

It will be necessary to incorporate Eq. (7) into Eq. (1c) in order to obtain a general expression for the tension in the swing cable. This process requires finding the derivative of Eq. (7).

The derivative of the function θ(t) with respect to time is obtained using the following standard results from differential calculus [2-9]:

$$\frac{d}{dt}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dt}, -\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}$$

and

$$\frac{d}{dx} sn(x) = cn(x) dn(x)$$

With the aid of these results, it is easy to verify that the instantaneous angular speed of the pendulum is given by:

$$\frac{d\theta}{dt} = \frac{2\omega \sin\left(\frac{\alpha}{2}\right) cn(\omega t)dn(\omega t)}{\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right) sn^2(\omega t)}}, -\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}. \quad (8)$$

Using the results of Eqs. (7) and (8) into Eq. (1c), the general expression for the tension in each cable that holds the seat becomes:

$$\frac{T}{\left(\frac{mg}{2}\right)} = \cos\left\{2 Arc\sin\left[\sin\left(\frac{\alpha}{2}\right) sn(\omega t)\right]\right\} + \left[2\sin\left(\frac{\alpha}{2}\right) \frac{cn(\omega t)dn(\omega t)}{\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right) sn^2(\omega t)}}\right]^2 \quad (9)$$

COMPARING THE ANGLES OF SWING

A graphical comparison of the angles of swing obtained, respectively, from the nonlinear and linear equations is shown in Figure 1. Six starting angles were chosen; and, for each, a solution was obtained using the linearised equation and another with the nonlinear equation. The initial angles used were α ≈ 10°, 30°, 60°, 90°, 120° and 150°. Plots of the corresponding variations of the angular positions of the pendulum with time are shown, with the solid lines representing the linear solution and the dashed lines the nonlinear (exact) solution [1][10][11].

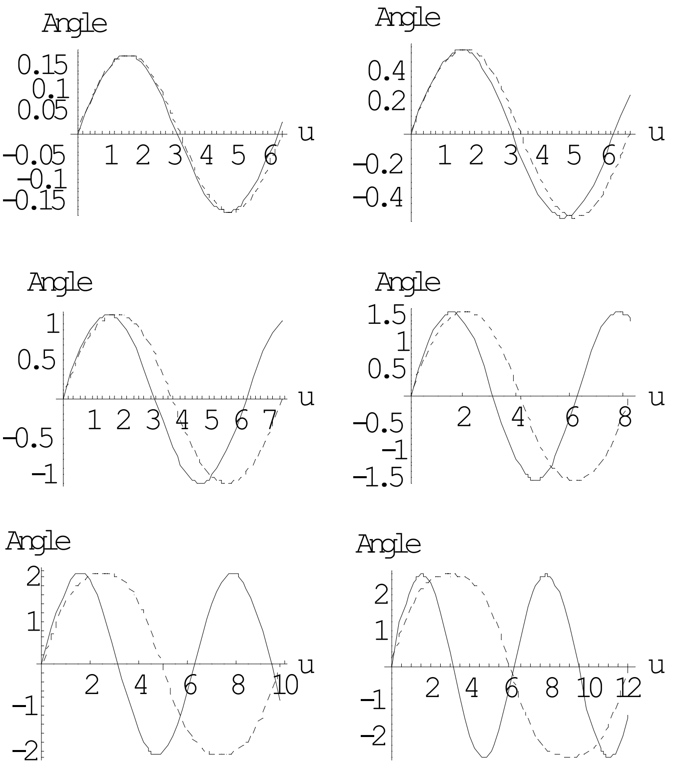

Figure 1: Swing angle vs. time, for α =10°, 30°, 60°, 90°, 120° and 150°. Solid lines show the approximate solution), while dashed lines show the exact solution [1].

COMPARING THE FORCES IN THE CABLE

Variations of the forces in the cable with the position of the pendulum were obtained from the nonlinear and linear equations, respectively, and are compared graphically in Figure 2 by using six starting angles: $\alpha = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{7\pi}{12}$, where the solid lines represent the *linear* (approximate) solution and the dashed lines the *nonlinear* (exact) solution.

From Figure 2, it can be seen that, as the initial amplitude α increases, so too does the discrepancy between the corresponding periods of motion and tensions in the cables that hold the seat.

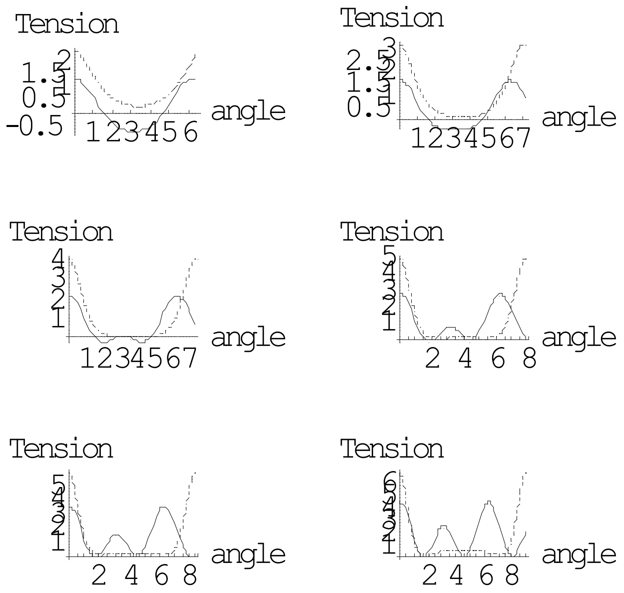

Figure 2: The tension in each cable that holds the seat. Eq. (6b, solid line) and Eq. (9, dashed line) are plotted over one complete cycle of the nonlinear solution. The initial angle α is a parameter ($\alpha = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{7\pi}{12}$, respectively).

Furthermore, for any initial angle, the maximum tension in each cable is reached when the swing hangs vertically under its supports (θ = 0). From Figure 2, it is clear that the linearised equation of motion underestimates the maximum tension in the cable at this point. The maximum tensions at θ = 0 were compared by computing their ratios as a function of the starting angle α.

The tension from Eq. (9) was divided by that obtained from Eq. (6b) and the result is illustrated in Figure 3. It is evident that the linearised equation underestimates the maximum tension in the cable and that it can do so by as much as 95%.

To demonstrate that, generally, underestimation of the magnitude of tension is a pattern that holds true for many positions of the swing besides θ =0, the ratio of the tensions is computed at a variety of locations of the pendulum. The results are plotted in Figure 4, for different positions of the pendulum.

It can be seen that each ratio is larger than unity, indicating that the tension from the nonlinear equation is larger than that from the linear equation. It can also be noted that, while all maximum tensions occur at one specific location of the pendulum, such is not the case for the ratio of tensions.

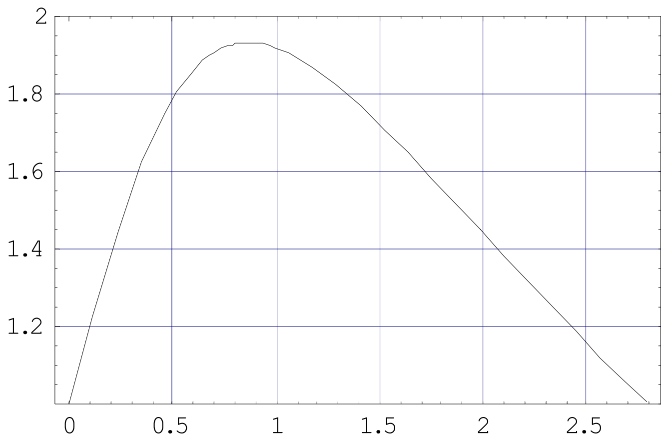

Figure 3: The ratio of maximum tensions in each cable was computed using Eq. (6b) and Eq. (9). Its variation with the initial angle α is plotted here. The maximum ratio was found to be 1.95 and occurred when the initial angle was α = 0.85 radian.

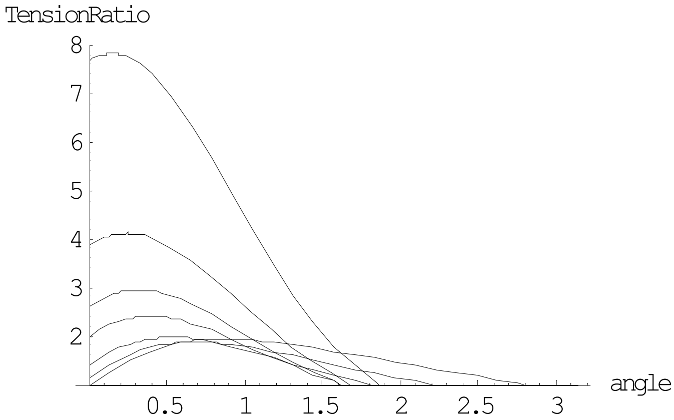

Figure 4: Each curve is a plot of the ratios of tensions in the cable, computed by dividing Eq. (9) by Eq. (6b), vs the initial angle α, assuming a fixed position of the pendulum. The positions used are: $\omega_n t = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{4.5\pi}{12}, \frac{5\pi}{12}, \frac{5.5\pi}{12}$.

CONCLUSIONS

It has been shown that the approximation $\sin\theta \approx \theta$, which is used to linearise the differential equation for the motion of the pendulum, introduces the following three kinds of errors in the kinematics of the motion:

- The magnitude of the period of oscillation;
- The magnitude of the swing angle;
- The phase of motion of the pendulum.

These errors were determined exactly and represented graphically [1].

A fourth error that is related to the dynamics of motion is identified in this paper and its magnitude estimated. The dynamic forces that arise in the supporting cables during the operation of a swing are determined under conditions of small angles and under those of large angles.

When the results are compared, one finds that the forces in the supporting cables that are computed based upon the linearised equation of motion are smaller than those obtained using the nonlinear equation itself and that the corresponding discrepancy increases with the magnitude of the initial angle given to the swing.

The effect of this discrepancy on design for strength and reliability is that the moments and forces that are used to design the beam and bolts that hold the seat of the swing are underestimated significantly when the driving force is linearised. Numerical experimentation indicates that, as a rule of thumb in design, forces computed using linearised motion must be increased by 100% in order to account for the effects of errors introduced by the linearisation of the governing differential equation of motion.

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